SOME DOUBLE BINOMIAL SUMS RELATED TO FIBONACCI, PELL AND GENERALIZED ORDER-k FIBONACCI NUMBERS

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ABSTRACT. We consider some double binomial sums related with the Fibonacci, Pell numbers and a multiple binomial sums related with the generalized order-k Fibonacci numbers. The Lagrange-Bürmann formula and other known techniques are used to prove them.

1. Introduction

The generating function of the Fibonacci numbers F_n is

$$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2}.$$

Similarly, the generating function of the Pell numbers P_n is

$$\sum_{n=0}^{\infty} P_n x^n = \frac{x}{1-2x-x^2}.$$

The generalized order-k Fibonacci numbers $f_n^{(k)}$ are defined by

$$f_n^{(k)} = \sum_{i=1}^{k} f_{n-i}^{(k)} \quad \text{for} \quad n > k$$

with initial conditions $f_j^{(k)} = 2^{j-1}$ for $1 \leq j \leq k$.

For example, when $k = 3$, the generalized Fibonacci numbers $f_n^{(3)}$ are reduced to the Tribonacci numbers T_n defined by

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}$$

with $T_1 = 1$, $T_2 = 2$ and $T_3 = 4$, for $n > 3$.

For these number sequences, we recall the combinatorial representations due to [2, 3, 5]:

$$\sum_{i=1}^{n} \binom{n-i}{i-1} = F_n, \tag{1.1}$$

$$\sum_{i=1}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2i+1} 2^r = P_n, \tag{1.2}$$

$$\sum_{i=0}^{n} \sum_{j=0}^{n} \binom{n-i}{j} \binom{n-j}{i} = F_{2n+3}. \tag{1.3}$$

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Among the formulas (1.1–1.3), the last formula seems to be different from first two identities just above since it includes double sums, see [2]. The authors of the above cited papers use a combinatorial approach to prove these results. For many similar identities, we refer to [6].

In this paper, we shall derive some new double binomial sums related with the Fibonacci, Pell and generalized order-k Fibonacci numbers and then use the Lagrange-Bürmann formula and well known other techniques to prove them.

The Lagrange-Bürmann formula is a very useful tool if one knows a series expansion for $y(x)$ but would like to obtain the series for x in terms of y. We recall the formula (for details see [1, 4]): Suppose a series for y in powers of x is required when $y = x\Phi(y)$. Assume that Φ is analytic in a neighborhood of $y = 0$ with $\Phi(0) \neq 0$. Then

$$x = y/\Phi(y) = \sum_{n=1}^{\infty} a_n y^n, \quad a_1 \neq 0.$$

Then the two (equivalent) version of the Lagrange(-Bürmann) inversion formula can be written as

$$F(y) = F(0) + \sum_{n=1}^{\infty} \frac{x^n}{n!} \left[\frac{d^{n-1}}{dy^{n-1}} \big(F'(y)\Phi^n(y)\big) \right]_{x=0}$$

or

$$\frac{F(y)}{1 - x\Phi'(y)} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \left[\frac{d^n}{dy^n} \big(F(y)\Phi^n(y)\big) \right]_{x=0}.$$

We would like to rephrase this using the notation of the "coefficient–of" operator:

$$\frac{F(y)}{1 - x\Phi'(y)} = \sum_{n=0}^{\infty} [y^n]\big(F(y)\Phi^n(y)\big) \cdot x^n;$$

we will use it in this form.

2. Double Binomial Sums

We start with a result related to Fibonacci numbers:

Theorem 1. *For $n > 0$,*

$$F_{4n-1} = \sum_{0 \le i,j \le n} \binom{n+i}{2j} \binom{n+j}{2i}.$$

Proof. We start from

$$[y^{2j}](1+y)^{n+i} = \binom{n+i}{2j}$$

and compute

$$\begin{aligned} S &= \sum_{i=0}^{n} (1+y)^{n+i} \binom{n+j}{2i} \\ &= \sum_{i \ge 0} (1+y)^{n+i/2} \binom{n+j}{i} \frac{1+(-1)^i}{2} \\ &= \left[\left(1 + \sqrt{1+y}\right)^{j+n} + \left(1 - \sqrt{1+y}\right)^{j+n} \right] \frac{(1+y)^n}{2}, \end{aligned}$$

here the desired sum takes the form:

$$\sum_{j=0}^{n}[y^{2j}]\left[\left(1+\sqrt{1+y}\right)^{j+n}+\left(1-\sqrt{1+y}\right)^{j+n}\right]\frac{(1+y)^n}{2}$$

$$=\sum_{j\ge 0}[y^{2j}]\left[\left(1+\sqrt{1+y}\right)^{j+n}+\left(1-\sqrt{1+y}\right)^{j+n}\right]\frac{(1+y)^n}{2}$$

$$=\sum_{j\ge 0}[y^{2j}]\left(1+\sqrt{1+y}\right)^{j+n}\frac{(1+y)^n}{2}+\sum_{j\ge 0}[y^{2j}]\left(1-\sqrt{1+y}\right)^{j+n}\frac{(1+y)^n}{2}$$

$$=\sum_{j\ge 0}[y^{j}]\left(1+\sqrt{1+y}\right)^{j/2+n}\frac{(1+y)^n}{2}\frac{1+(-1)^j}{2}+\sum_{j\ge 0}[y^{2j}]\left(1-\sqrt{1+y}\right)^{j+n}\frac{(1+y)^n}{2}.$$

Let us consider the first sum:

$$\sum_{j\ge 0}[y^{j}]\left(1+\sqrt{1+y}\right)^{j/2+n}(1+y)^n.$$

This is of the form

$$\sum_{j\ge 0}[y^{j}]F(y)\Phi(y)^j$$

with

$$F(y)=\left(1+\sqrt{1+y}\right)^{n}(1+y)^n \quad\text{and}\quad \Phi(y)=\sqrt{1+\sqrt{1+y}}.$$

The Lagrange-Bürmann formula can now be applied to this sum. The general formula is given by

$$\sum_{j\ge 0}[y^{j}]F(y)\Phi(y)^j\cdot x^j=\frac{F(y)}{1-x\Phi'(y)}.$$

We need the instance $x=1$ here, and the variables x and y are linked via $y=x\Phi(y)$. Notice that $\Phi(y)$ must be a power series in y with a constant term different from zero. Therefore

$$y=\frac{1+\sqrt{5}}{2},\quad F(\alpha)=\left(\frac{7+3\sqrt{5}}{2}\right)^n,$$

$$\Phi'(\alpha)=\frac{3-\sqrt{5}}{8},\quad \frac{1}{1-\Phi'(\alpha)}=2\left(1-\frac{1}{\sqrt{5}}\right).$$

So our evaluation is

$$2\left(1-\frac{1}{\sqrt{5}}\right)\left(\frac{7+3\sqrt{5}}{2}\right)^n.$$

The second term is

$$\sum_{j\ge 0}[y^{j}]\left(1+\sqrt{1+y}\right)^{j/2+n+1/2}(1+y)^n(-1)^j.$$

This is the instance $x=-1$, which translates to $y=-1$ and so the third term is

$$\frac{F(-1)}{1+\Phi'(-1)}=0.$$

The last sum is

$$\sum_{j\ge0}[y^{2j}]\left(1-\sqrt{1+y}\right)^{j+n}(1+y)^n=\sum_{j\ge0}[y^{2j}]y^{j+n}\left(\frac{1-\sqrt{1+y}}{y}\right)^{j+n}(1+y)^n$$
$$=\sum_{j\ge0}[y^{j}]y^{n}\left(\frac{1-\sqrt{1+y}}{y}\right)^{j+n}(1+y)^n.$$

This is again of the form

$$\sum_{j\ge0}[y^j]F(y)\Phi(y)^j$$

with

$$F(y)=\left(1-\sqrt{1+y}\right)^n(1+y)^n \quad\text{and}\quad \Phi(y)=\frac{1-\sqrt{1+y}}{y}.$$

We need the instance $x=1$ here, and the link is

$$y=x\left(\frac{1-\sqrt{1+y}}{y}\right),$$

which means

$$y=\frac{1-\sqrt5}{2},\quad F(\beta)=\left(\frac{7-3\sqrt5}{2}\right)^n \quad\text{and}\quad \frac{1}{1-\Phi'(\alpha)}=1+\frac{1}{\sqrt5}.$$

So our evaluation is

$$\left(1+\frac{1}{\sqrt5}\right)\left(\frac{7-3\sqrt5}{2}\right)^n.$$

Altogether

$$\left[\left(1-\frac{1}{\sqrt5}\right)\left(\frac{7+3\sqrt5}{2}\right)^n+\left(1+\frac{1}{\sqrt5}\right)\left(\frac{7-3\sqrt5}{2}\right)^n\right]\frac12=\frac{\alpha^{4n-1}-\beta^{4n-1}}{\sqrt5}=F_{4n-1},$$

as desired. ∎

Theorem 2. *For $n>0$,*

$$F_{4n+1}=\sum_{1\le i,j\le n+1}\binom{n+i}{2j-1}\binom{n+j}{2i-1}.$$

Proof. Since

$$\left[y^{2j-1}\right](1+y)^{n+i}=\binom{n+i}{2j-1}$$

and

$$S=\sum_{i=1}^{n+1}(1+y)^{n+i}\binom{n+j}{2i-1}$$
$$=\sum_{i\ge0}(1+y)^{n+(i+1)/2}\binom{n+j}{i}\frac{1-(-1)^i}{2}$$
$$=\left[\left(1+\sqrt{1+y}\right)^{j+n}-\left(1-\sqrt{1+y}\right)^{j+n}\right]\frac{(1+y)^{n+1/2}}{2},$$

here the desired sum takes the form:

$$\begin{aligned}
&\sum_{j=1}^{n+1}[y^{2j-1}]\left[\left(1+\sqrt{1+y}\right)^{j+n}-\left(1-\sqrt{1+y}\right)^{j+n}\right]\frac{(1+y)^{n+1/2}}{2}\\
&=\sum_{j\ge 1}[y^{2j-1}]\left[\left(1+\sqrt{1+y}\right)^{j+n}-\left(1-\sqrt{1+y}\right)^{j+n}\right]\frac{(1+y)^{n+1/2}}{2}\\
&=\sum_{j\ge 1}[y^{2j-1}]\left(1+\sqrt{1+y}\right)^{j+n}\frac{(1+y)^{n+1/2}}{2}\\
&\quad-\sum_{j\ge 1}[y^{2j-1}]\left(1-\sqrt{1+y}\right)^{j+n}\frac{(1+y)^{n+1/2}}{2}\\
&=\sum_{j\ge 0}[y^{j}]\left[\left(1+\sqrt{1+y}\right)^{j/2+n+1/2}\right]\frac{(1+y)^{n+1/2}}{2}\frac{1-(-1)^j}{2}\\
&\quad-\sum_{j\ge 1}[y^{2j-1}]\left(1-\sqrt{1+y}\right)^{j+n}\frac{(1+y)^{n+1/2}}{2}.
\end{aligned}$$

Let us start with one term in the above sum:

$$\sum_{j\ge 0}[y^j]\left(1+\sqrt{1+y}\right)^{j/2+n+1/2}(1+y)^{n+1/2}.$$

This is of the form

$$\sum_{j\ge 0}[y^j]F(y)\Phi(y)^j$$

with

$$F(y)=\left(1+\sqrt{1+y}\right)^{n+1/2}(1+y)^{n+1/2}\quad\text{and}\quad\Phi(y)=\sqrt{1+\sqrt{1+y}}.$$

This is the instance $x=1$, which translates to

$$y=\frac{1+\sqrt{5}}{2},\quad F(\alpha)=\alpha^{4n+2}$$

and

$$\Phi'(\alpha)=\frac{3-\sqrt{5}}{8},\quad \frac{1}{1-\Phi'(\alpha)}=2\left(1-\frac{1}{\sqrt{5}}\right).$$

So our evaluation is:

$$2\left(1-\frac{1}{\sqrt{5}}\right)\alpha^{4n+2}.$$

The second term is

$$\sum_{j\ge 0}[y^j]\left(1+\sqrt{1+y}\right)^{j/2+n+1/2}(1+y)^{n+1/2}(-1)^j.$$

This is the instance $x=-1$, which translates to $y=-1$ and so the second term is

$$\frac{F(-1)}{1+\Phi'(-1)}=0.$$

Finally the last term is of the form:

$$\begin{aligned}
&\sum_{j\ge1}[y^{2j-1}]\Big(1-\sqrt{1+y}\Big)^{j+n}(1+y)^{n+1/2}\\
&\qquad=\sum_{j\ge1}[y^{2j-1}]y^{j+n}\left(\frac{1-\sqrt{1+y}}{y}\right)^{j+n}(1+y)^{n+1/2}\\
&\qquad=\sum_{j\ge0}[y^{j}]y^{n+1}\left(\frac{1-\sqrt{1+y}}{y}\right)^{j+n}(1+y)^{n+1/2}.
\end{aligned}$$

This is of the form:

$$\sum_{j\ge0}[y^j]F(y)\Phi(y)^j$$

with

$$F(y)=\Big(1-\sqrt{1+y}\Big)^{n}(1+y)^{n+\frac12}y \quad\text{and}\quad \Phi(y)=\frac{1-\sqrt{1+y}}{y}.$$

This is the instance $x=1$, which translates to $y=\beta=\frac{1-\sqrt5}{2}$. Thus

$$F(\beta)=-\beta^{4n+2},\quad \Phi'(\beta)=-\frac{1-\sqrt5}{4},\quad \frac{F(\beta)}{1-\Phi'(\beta)}=-\left(1+\frac{1}{\sqrt5}\right)\beta^{4n+2}.$$

So our evaluation is

$$\left[\left(1-\frac{1}{\sqrt5}\right)\alpha^{4n+2}+\left(1+\frac{1}{\sqrt5}\right)\beta^{4n+2}\right]\frac12=F_{4n+1},$$

as claimed. ∎

Theorem 3. *For $n>0$,*

$$\begin{aligned}
F_{4n}&=\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{n+i}{2j-1}\binom{n+j}{2i},\\
F_{4n-3}&=\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{n+i}{2j+1}\binom{n+j}{2i+1}.
\end{aligned}$$

Again by using the Lagrange-Bürmann formula, Theorem 3 can be similarly proved.

Theorem 4. *For $n>0$,*

$$\frac{F_{2n+2}+F_{n+1}}{2}=\sum_{0\le i,j\le n}\binom{n-i}{2j}\binom{n-2j}{i}.$$

Proof. First, we replace i by $n-i$ and get

$$\sum_{0\le 2j\le i\le n}\binom{i}{2j}\binom{n-2j}{i-2j}.$$

Now we compute the generating function of it:

$$\begin{aligned}\sum_{n\geq 0} z^n \sum_{0\leq 2j\leq i\leq n} \binom{i}{2j}\binom{n-2j}{i-2j} &= \sum_{0\leq 2j\leq i} \binom{i}{2j} \frac{z^i}{(1-z)^{i+1-2j}}\\ &= \sum_{j\geq 0} \frac{z^{2j}(1-z)^{2j}}{(1-2z)^{1+2j}} = \frac{1-2z}{(1-z-z^2)(1-3z+z^2)}\\ &= \frac{1}{2}\frac{1}{1-z-z^2} + \frac{1}{2}\frac{1}{1-3z-z^2},\end{aligned}$$

which is the generating function of the numbers $(F_{2n+2}+F_{n+1})/2$. ∎

The following results are similar:

Theorem 5. *For* $n>0$,

$$F_{2n} = \sum_{i=1}^{n}\sum_{j=1}^{n} \binom{n-i}{j-1}\binom{n-j}{i-1},$$

$$F_{2n-1} = \sum_{0\leq j\leq i\leq n} \binom{n}{i-j}\binom{n-i}{j}.$$

Theorem 6. *For* $n>0$,

$$F_{2n}+1 = \sum_{i=0}^{n} F_{2i-1} = \sum_{0\leq i\leq j\leq n} \binom{n-i}{j}\binom{j}{2i}. \tag{2.1}$$

Proof. Multiplying the right hand side of (2.1) by z^n and summing over n, we get

$$\begin{aligned}S = \sum_{n\geq 0} z^n \sum_{0\leq i\leq j\leq n} \binom{n-i}{j}\binom{j}{2i} &= \sum_{0\leq i\leq j}\sum_{h\geq 0} z^{h+i+j}\binom{h+j}{j}\binom{j}{2i}\\ &= \sum_{0\leq i\leq j}\binom{j}{2i} z^{i+j}\sum_{h\geq 0} z^h \binom{h+j}{j} = \sum_{0\leq 2i\leq j}\binom{j}{2i} z^{i+j}\frac{1}{(1-z)^{j+1}}\\ &= \sum_{i\geq 0}\frac{z^{3i}}{(1-2z)^{2i+1}} = \frac{1-2z}{(1-z)(1-3z+z^2)} = \frac{z}{1-3z+z^2} + \frac{1}{1-z},\end{aligned}$$

which is the generating function of the numbers $F_{2n}+1$. ∎

For the Pell numbers, we give the following result:

Theorem 7. *For* $n\geq 0$,

$$P_{n+1} = \sum_{0\leq i\leq j\leq n} \binom{n-i}{j}\binom{j}{i}. \tag{2.2}$$

Proof. Multiplying the right hand side of (2.2) by z^n and summing over n, we get

$$\begin{aligned}S = \sum_{n\geq 0} z^n \sum_{0\leq i\leq j\leq n} \binom{n-i}{j}\binom{j}{i} &= \sum_{0\leq i\leq j}\sum_{h\geq 0} z^{h+i+j}\binom{h+j}{j}\binom{j}{i}\\ &= \sum_{0\leq i\leq j}\binom{j}{i} z^{i+j}\sum_{h\geq 0} z^h\binom{h+j}{j} = \sum_{0\leq i\leq j}\binom{j}{i} z^{i+j}\frac{1}{(1-z)^{j+1}}\end{aligned}$$

$$= \sum_{0\le i\le j} \frac{z^j}{(1-z)^{j+1}}\binom{j}{i}z^i = \sum_{j\ge 0}\frac{z^j}{(1-z)^{j+1}}(1+z)^j$$
$$= \frac{1}{1-z}\frac{1}{1-\dfrac{z(1+z)}{1-z}} = \frac{1}{1-2z-z^2}.$$

This is the generating function of the numbers P_{n+1}. ∎

Now we give a double sum for the Tribonacci numbers:

Theorem 8. *For $n \ge 0$,*

$$T_n = \sum_{0\le j\le i\le n}\binom{n-i}{i-j}\binom{i-j}{j}.$$

Proof. Consider

$$\sum_{n\ge 0} T_n z^n = \sum_{0\le j\le i\le n} z^n \binom{n-i}{i-j}\binom{i-j}{j} = \sum_{0\le j\le i} z^i\binom{i-j}{j}\sum_{h\ge 0} z^h\binom{h}{i-j}$$
$$= \sum_{0\le j\le i} z^i\binom{i-j}{j}\frac{z^{i-j}}{(1-z)^{i-j+1}} = \sum_{j\ge 0}\sum_{h\ge 0} z^{h+j}\binom{h}{j}\frac{z^h}{(1-z)^{h+1}}.$$

Let $t = \dfrac{z^2}{1-z}$, and we continue

$$\sum_{n\ge 0} T_n z^n = \frac{1}{1-z}\sum_{0\le j} z^j \sum_{h\ge 0}\binom{h}{j}t^h = \frac{1}{1-z}\sum_{0\le j} z^j \frac{t^j}{(1-t)^{j+1}}$$
$$= \frac{1}{1-z}\frac{1}{1-t}\frac{1}{1-\dfrac{zt}{1-t}} = \frac{1}{1-z}\frac{1}{1-t-zt}$$
$$= \frac{1}{1-z}\frac{1}{1-\dfrac{z^2}{1-z}-\dfrac{z^3}{1-z}} = \frac{1}{1-z-z^2-z^3},$$

which is the generating function of the Tribonacci numbers, as expected. So the proof is complete. ∎

By using the same proof method as in Theorem 8, we get a more general result:

Theorem 9. *For $n > 0$,*

$$f_n^{(k)} = \sum_{0\le i_k\le\cdots\le i_1\le n}\binom{n-i_1}{i_1-i_2}\binom{i_1-i_2}{i_2-i_3}\cdots\binom{i_{k-1}-i_k}{i_k}$$

where $f_n^{(k)}$ is the n-th generalized order-k Fibonacci number.

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